Prova 2 - Turma 2 (tarde) - 21/08/2024

1) b) $f(x,y) = xy - x^3 - y^2$

$\nabla f(x,y) = (y - 3x^2, x - 2y)$

$\nabla f(x,y) = (0,0) \Leftrightarrow \begin{cases} y - 3x^2 = 0 \\ x - 2y = 0 \end{cases}$

$x - 2y = 0 \Rightarrow x = 2y$

$y - 3x^2 = 0 \Rightarrow y - 12y^2 = 0 \Rightarrow y(1 - 12y) = 0$

$y = 0$ ou $y = 1/12$

- $y = 0 \Rightarrow x = 0$
- $y = 1/12 \Rightarrow x = 1/6$

Pontos críticos: $(0,0)$, $(1/6, 1/12)$

$$H(x,y) = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix}$$

$$= \begin{vmatrix} -6x & 1 \\ 1 & -2 \end{vmatrix}$$

- $H(0,0) = \begin{vmatrix} 0 & 1 \\ 1 & -2 \end{vmatrix} = -1$

$(0,0)$ é ponto de sela

- $H(1/6, 1/12) = \begin{vmatrix} -1 & 1 \\ 1 & -2 \end{vmatrix} = 2 - 1 = 1$

$f(1/6, 1/12)$ é valor de máximo local

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2) b) $f(x,y) = xy + 4x + 2$

$F(x,y,z) = xy + 4x + 2 - z$

$F(x,y,z) = 0$

Plano tangente em $P(a,b,c)$

$\nabla F(a,b,c)\,(x-a,\, y-b,\, z-c) = 0$

$(b+4,\, a,\, -1)(x-a,\, y-b,\, z-c) = 0$

$\nabla F(a,b,c) \parallel v,$

onde v é o vetor normal ao plano xy $(z=0)$.

$v = (0,0,1)$

logo

$(b+4,\, a,\, -1) \parallel (0,0,1)$

Daí

$(b+4,\, a,\, -1) = -1\,(0,0,1) \Rightarrow$

$b+4=0 \Rightarrow b=-4$

$a=0$

$\therefore$ a equação do plano tangente procurado será:

$(b+4, a, -1)(x-a, y-b, z-c)=0$

$(0,0,-1)(x-0, y+4, z-f(0,-4))=0$

$-z+f(0,-4)=0$

$-z+2=0 \Rightarrow \boxed{z=2}$

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3) $D=\{(x,y)\in\mathbb{R}^2 : x^2+y^2\leq 1\}$

$T(x,y)=x^2+2y^2-x$

- No interior do disco

$\nabla T(x,y)=(2x-1, 4y)$

$\nabla T(x,y)=(0,0) \Rightarrow \begin{cases} 2x=1 \Rightarrow x=1/2 \\ 4y=0 \Rightarrow y=0 \end{cases}$

Ponto crítico: $(1/2, 0)\in D$

$T(1/2, 0) = \frac{1}{4} - \frac{1}{2} = -\frac{1}{4}$

- Na fronteira: $g(x,y) = x^2 + y^2$

$$\begin{cases} \nabla T(x,y) = \lambda \nabla g(x,y) \\ g(x,y) = 1 \end{cases}$$

$$\begin{cases} (2x-1, 4y) = \lambda (2x, 2y) \\ x^2 + y^2 = 1 \end{cases}$$

$$\begin{cases} 2x - 1 = 2\lambda x \\ 4y = 2\lambda y \\ x^2 + y^2 = 1 \end{cases}$$

$4y - 2\lambda y = 0 \Rightarrow 2y(2 - \lambda) = 0$

$\Rightarrow y = 0$ ou $\lambda = 2$

- $y = 0$

$x^2 = 1 \Rightarrow x = \pm 1$

- $\lambda = 2$

$2x - 1 = 4x \Rightarrow 2x = -1 \Rightarrow x = -1/2$

$x^2 + y^2 = 1 \Rightarrow \frac{1}{4} + y^2 = 1 \Rightarrow$

$y^2 = \frac{3}{4} \Rightarrow y = \pm \frac{\sqrt{3}}{2}$

Pontos:

$(1, 0), (-1, 0), (-1/2, \sqrt{3}/2), (-1/2, -\sqrt{3}/2)$

$T(1,0) = 0$

$T(-1,0) = 2$

$T(-1/2, \sqrt{3}/2) = \frac{1}{4} + \frac{3}{2} + \frac{1}{2} = \frac{9}{4}$

$T(-1/2, -\sqrt{3}/2) = \frac{9}{4}$

Portanto,

pontos de maior aquecimento

$$\left(\frac{-1}{2}, \frac{\sqrt{3}}{2}\right), \left(\frac{-1}{2}, \frac{-\sqrt{3}}{2}\right)$$

ponto de menor aquecimento

$$\left(\frac{1}{2}, 0\right)$$

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4) $f(x,y) = \sqrt{4-x^2-y^2}$

a) $\nabla f(x,y) = \left(\frac{-x}{\sqrt{4-x^2-y^2}}, \frac{-y}{\sqrt{4-x^2-y^2}}\right)$

Direção da variação máxima:

$$\nabla f(0,1) = \left(0, \frac{-1}{\sqrt{3}}\right)$$

Taxa de variação máxima

$$\|\nabla f(0,1)\| = \sqrt{0^2 + \frac{1}{3}} = \frac{1}{\sqrt{3}}$$

b) $D_v f(0,1) = \nabla f(0,1) \cdot \frac{v}{\|v\|}$

$$= \left(0, -\frac{1}{\sqrt{3}}\right)(2,2) \cdot \frac{1}{\sqrt{8}}$$

$$= \frac{1}{2\sqrt{2}}\left(-\frac{2}{\sqrt{3}}\right) = -\frac{1}{\sqrt{6}}$$

c) $$f_{xx}(x,y) = \frac{-\sqrt{4-x^2-y^2} + x \cdot \left(\frac{-x}{\sqrt{4-x^2-y^2}}\right)}{4-x^2-y^2}$$

$$= \frac{-(4-x^2-y^2) - x^2}{(4-x^2-y^2)\sqrt{4-x^2-y^2}}$$

$$f_{xx}(0,1) = \frac{-3-0}{3\sqrt{3}} = -\frac{1}{\sqrt{3}}$$

$$f_{xy}(x,y) = \frac{-xy}{\sqrt{4-x^2-y^2}\,(4-x^2-y^2)}$$

$$f_{xy}(0,1) = 0$$

$$f_{yy}(x,y) = \frac{-(4-x^2-y^2)-y^2}{(4-x^2-y^2)\sqrt{4-x^2-y^2}}$$

$$f_{yy}(0,1) = \frac{-3-1}{3\sqrt{3}} = \frac{-4}{3\sqrt{3}}$$