TRUNCATED WEYL MODULES: A BRIEF INTRODUCTION AND SOME RESULTS

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Introduction

Chari and Pressley introduced a family of finite dimensional representations called Weyl modules, for the affine Kac Moody algebra. Some different authors gave similar definitions for Weyl modules for different classes of algebras. Our work is concerned with the study of the structure of Weyl modules for truncated current algebras, the so called truncated Weyl modules.

Chari, Fourier and Sagaki conjectured that any truncated Weyl module is isomorphic to some fusion product of irreducible modules. Kus and Littelman, and Ravinder proved special cases of this conjecture.

We present some modules which have been used as important tools to study local Weyl modules, we give a λ-compatible $|R_+|$-tuple of partitions ξ, such that, the CV-module $V(\xi)$ is isomorphic to the truncated Weyl module $W_N(\lambda)$ and we prove a special case of the Chari-Fourier-Sagaki conjecture.

Preliminaries

Let $\mathfrak{g}$ be a finite-dimensional simple Lie algebra over the complex numbers and denote by $\mathcal{U}(\mathfrak{g})$ its universal enveloping algebra. Fix $\mathfrak{h}\subset\mathfrak{g}$ a Cartan subalgebra of $\mathfrak{g}$. Let R_+ be the set of positive roots corresponding to the choice of $\mathfrak{h}$, $\prod=\{\alpha_1,\dots,\alpha_n\}$ the simple roots, $\omega_1,\dots,\omega_n$ the corresponding fundamental weights, $I=\{1,\dots,n\}$. Denote by $\mathcal{P}^+=\sum_i\mathbb{Z}_{\geq 0}\omega_i$, the dominant weight lattice.

The highest root will be denoted by θ. Given $\eta=\sum_i a_i\alpha_i\in Q$ and $i\in I$, set

$$\mathrm{ht_i}(\eta)=\mathrm{a_i}\quad\text{and}\quad \mathrm{ht}(\eta)=\sum_{\mathrm{i}\in\mathrm{I}}\mathrm{ht_i}(\eta).$$

Let e_α, f_α be the Chevalley generators and consider the following notation $e_{\alpha_i}=e_i$, $f_{\alpha_i}=f_i$, $h_{\alpha_i}=h_i$. If $\alpha\in R$ then $\mathfrak{g}_\alpha$ is the corresponding root space of $\mathfrak{g}$. Then

$$\mathfrak{g}=\mathfrak{n}^+\oplus\mathfrak{h}\oplus\mathfrak{n}^-,$$

where $\mathfrak{n}^\pm=\bigoplus_{\alpha\in R_+}\mathfrak{g}_{\pm\alpha}$.

Local Weyl Modules

The Lie algebra $\mathfrak{g}\otimes\mathbb{C}[t]$ is called **current Lie algebra** over $\mathfrak{g}$ and it is denoted by $\mathfrak{g}[t]$.

Let V be a $\mathfrak{g}$-module. Given $a\in\mathbb{C}$, let $ev_a:\mathfrak{g}[t]\to\mathfrak{g}$ be the evaluation map $x\otimes f(t)\mapsto f(a)x$, which is a Lie algebra homomorphism. Then, we can consider V as the $\mathfrak{g}[t]$-module obtained by pulling-back the action of $\mathfrak{g}$ to $\mathfrak{g}[t]$ via ev_a. We denote the $\mathfrak{g}[t]$-module by V_a.

The Weyl modules were introduced as modules having the universal property: *"any finite dimensional highest weight module which is generated by a one dimensional highest weight space is a quotient of a Weyl module."*.

For $\lambda\in\mathcal{P}^+$, the **graded local Weyl module** $W(\lambda)$ is the cyclic $\mathfrak{g}[t]$-module generated by a nonzero vector w_λ satisfying the following defining relations

$$\mathfrak{n}^+[t]w_\lambda=0;\ \ (h\otimes t^k)w_\lambda=\delta_{0,k}\lambda(h)w_\lambda,\ \forall\, h\in\mathfrak{h};$$
$$(f_\alpha\otimes 1)^{\lambda(h_\alpha)+1}w_\lambda=0,\ \forall\,\alpha\in R_+.$$

And the **truncated Weyl module** $W_N(\lambda)$ is a $\mathfrak{g}[t]$-module generated by a nonzero vector v_λ with the following defining relations

$$\mathfrak{n}^+[t]_N v_\lambda=0;\ \ (h\otimes t^k)v_\lambda=\delta_{0,k}\lambda(h)v_\lambda,\ \forall\, h\in\mathfrak{h};$$
$$(f_\alpha\otimes 1)^{\lambda(h_\alpha)+1}v_\lambda=0=(f_\alpha\otimes t^N)v_\lambda,\ \forall\,\alpha\in R_+.$$

Given $\lambda\in\mathcal{P}^+$, consider the local Weyl module $W(\lambda)$ generated by a nonzero vector w_λ. Let $L_{N,\lambda}$ be the $\mathfrak{g}[t]$-submodule of $W(\lambda)$ generated by the elements $(x\otimes t^N)w_\lambda$, for all $x\in\mathfrak{g}$.

Proposition: As $\mathfrak{g}[t]$-modules we have

$$W_N(\lambda)\simeq\frac{W(\lambda)}{L_{N,\lambda}}.$$

Chari-Venkatesh Modules

In [2], Chari and Venkatesh introduced a family of indecomposable finite-dimensional graded modules for the current algebra associated to a simple Lie algebra, which are now called *Chari-Venkatesh (CV)* modules. These modules are indexed by an $|R_+|$-tuple of partitions $\xi=(\varepsilon(\alpha))$.

Given $\lambda\in\mathcal{P}^+$, we say that $\xi=(\varepsilon(\alpha))_{\alpha\in R_+}$ is a λ-**compatible** $|R_+|$-tuple of partitions, if

$$\varepsilon(\alpha)=(\varepsilon(\alpha)_1\geq\ldots\geq\varepsilon(\alpha)_s\geq\ldots\geq 0),\ \ |\varepsilon(\alpha)|=\sum_{j\geq 1}\varepsilon(\alpha)_j=\lambda(h_\alpha),\ \forall\,\alpha\in R_+.$$

From now on, $\xi=(\varepsilon(\alpha))_{\alpha\in R_+}$ is always a λ-compatible $|R_+|$-tuple of partitions.

Definition

The **Chari-Venkatesh module** $V(\xi)$ is the graded quotient of $W(\lambda)$ by the submodule generated by the graded elements

$$\{(e_\alpha\otimes t)^s(f_\alpha\otimes 1)^{s+r}w_\lambda:\alpha\in R_+, s,r\in\mathbb{Z}_{>0}, s+r\geq 1+rk+\sum_{j\geq k+1}\varepsilon(\alpha)_j,\ \text{ for some }\ k\in\mathbb{Z}_{>0}\}.$$

Fusion Product

Let V^i be a finite dimensional cyclic graded $\mathfrak{g}[t]$-module generated by v_i, for $1\leq i\leq k$, and let $a_1,\dots,a_k$ be distinct complex numbers. Set

$$V=V^1_{a_1}\otimes\ldots\otimes V^k_{a_k},$$

which is a cyclic $\mathfrak{g}[t]$-module generated by $v=v_1\otimes\ldots\otimes v_k$ and consider

$$\mathcal{U}(\mathfrak{g}[t])[s]:=span\{(x_1\otimes t^{r_1})\ldots(x_k\otimes t^{r_k}):\,k\geq 1,\,x_i\in\mathfrak{g},\,r_i\in\mathbb{Z}_{\geq 0},\sum r_i=s\}.$$

We define a filtration F^rV, $r\in\mathbb{Z}_{\geq 0}$ by

$$F^rV=\sum_{0\leq s\leq r}\mathcal{U}(\mathfrak{g}[t])[s]v.$$

The associated graded module $grV=\bigoplus_{r\geq 0}\frac{F^rV}{F^{r-1}V}$ becomes a cyclic $\mathfrak{g}[t]$-module with action given by $(x\otimes t^s)(w+F^{r-1}V)=(x\otimes t^s)w+F^{r+s-1}V$, for all $x\in\mathfrak{g}$, $w\in F^rV$, $r,s\in\mathbb{Z}_{\geq 0}$.

The module grV is called the **fusion product** of $V^1,\dots,V^k$ with respect to the parameters $a_1,\dots,a_k$ and is denoted $V^1_{a_1}*\ldots*V^k_{a_k}$.

Proposition: Let $\mathfrak{g}$ be a simple Lie algebra and $\lambda=\lambda_1+\ldots+\lambda_N$, $\lambda_i\in\mathcal{P}^+$, for all $1\leq i\leq N$. There exists a surjective homomorphism

$$W_N(\lambda)\twoheadrightarrow V_{a_1}(\lambda_1)*\ldots*V_{a_N}(\lambda_N).$$

Chari-Venkatesh and Truncated Weyl Modules

Theorem

Let $\mathfrak{g}$ be a simple complex Lie algebra and $\lambda\in\mathcal{P}^+$. Given $N\in\mathbb{Z}_{>0}$, for each $\alpha\in R_+$, let a_α,b_α be the unique non negative integers such that

$$\lambda(h_\alpha)=Na_\alpha+b_\alpha,\ 0\leq b_\alpha<N.$$

Consider $\xi=(\varepsilon(\alpha))_{\alpha\in R_+}$ a λ-compatible $|R_+|$-tuple of partitions given by

$$\varepsilon(\alpha)=(\varepsilon(\alpha)_1\geq\ldots\varepsilon(\alpha)_N\geq 0)$$

with

$$\varepsilon(\alpha)_1=\ldots=\varepsilon(\alpha)_{b_\alpha}=a_\alpha+1\ \textit{ and }\ \varepsilon(\alpha)_{b_\alpha+1}=\ldots=\varepsilon(\alpha)_N=a_\alpha.$$

Then there exists an isomorphism of $\mathfrak{g}[t]$-modules

$$V(\xi)\simeq W_N(\lambda).$$

Chari-Fourier-Sagaki conjecture

The following conjecture was formulated by Chari, Fourier and Sagaki:

Conjecture

Let $N\in\mathbb{Z}_{>0},\lambda\in P^+$, and suppose $\boldsymbol{\lambda}=(\lambda_1,\dots,\lambda_N)$ is a maximal element of $P^+(\lambda,N)$. If $N\leq|\lambda|$, then $W_N(\lambda)\cong V(\lambda_1)*\cdots*V(\lambda_N)$ as graded $\mathfrak{g}[t]$-modules.

The Chari-Fourier-Sagaki conjecture has been proved in the following special cases:

- for simply laced $\mathfrak{g}$, $\lambda=m\theta$ for some $m\geq 0$, and $N=|\lambda|$ by Ravinder;
- for $\mathfrak{g}$ of type A, $N=2$, and $\lambda=m\omega_i$ for some $i\in I$ by Fourier;
- for $\lambda=N\mu+\nu$ with $\mu\in P^+_{sym}$ and ν minuscule by Kus and Littelmann.

Our main result generalizes and provides an alternate proof for item (2) above:

Theorem

Suppose $i\in I$ satisfies $\mathrm{ht_i}(\theta)=1$. Then, Chari-Fourier-Sagaki Conjecture holds for $\lambda=m\omega_i$ for all $m\geq 0$.

References

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